Solutions to Final

1. (20 points) Find both the complex and the sine/cosine Fourier series for the function

$$f(x) = |\sin x|$$

Hint: The period of $f(x)$ is *not* the same as the period of $\sin x$.

Answer: The period here is π, not 2π as it is for $\sin x$. Therefore the basis functions are $\sin 2nx$ and $\cos 2nx$, if we first do the sine/cosine series. The sine components are zero, because $f(x)$ is an even function:

$$b_n = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \sin 2nx\, |\sin x| = 0$$

The cosine components:

$$a_n = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \cos 2nx\, |\sin x| = \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} dx\, \cos 2nx \sin x = \frac{4}{\pi} \frac{1}{1-4n^2}$$

This works for all n, including $n = 0$. Therefore

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

The complex series is found by using

$$c_n = \frac{a_n - ib_n}{2} = \frac{2}{\pi} \frac{1}{1-4n^2}$$

So that

$$f(x) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i2nx}}{4n^2 - 1}$$

2. (20 points) You have a matrix

$$M = \begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(a) Find the eigenvalues and a set of *orthonormal* eigenvectors for M.

Answer: Following the usual procedure:

$$\begin{vmatrix} 1-\lambda & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)[\lambda(\lambda-1)-2=0 \quad \Rightarrow \quad \lambda=-1,2,2$$

Looking at the matrix, we can see that one of the eigenvectors corresponding to $\lambda = 2$ is just

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This is already normalized. The other two eigenvectors, to be orthogonal to this, can just have their third row equal to zero. This will save the trouble of subtracting out parallel components and so forth. After normalization, we end up with

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ 0 \end{pmatrix} \quad , \quad \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \frac{i}{\sqrt{2}} \\ 0 \end{pmatrix}$$

All of these are orthonormal.

(b) You have an arbitrary vector

$$|\psi\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where a, b, and c are complex numbers. Calculate the "expectation value"

$$\langle M\rangle_\psi = \langle\psi|M|\psi\rangle$$

and show that $\langle M\rangle_\psi$ is always a real number, no matter what a, b, and c are. (*Hint:* Don't forget how to get the dual vector $\langle\psi|$.)

Answer: It's a straightforward calculation:

$$\begin{aligned}\langle\psi|M|\psi\rangle &= \begin{pmatrix} a^* & b^* & c^* \end{pmatrix}\begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= a^*a + i\sqrt{2}(b^*a - a^*b) + 2c^*c\end{aligned}$$

Note that z^*z is a real number for any complex number z. The troublesome part is $i(b^*a - a^*b)$. But for that we notice that $(b^*a)^* = a^*b$. And for any complex number z, $z - z^*$ is an *imaginary* number. Therefore $i(b^*a - a^*b)$ has to be real. Hence $\langle M\rangle_\psi$ is always a real number.

3. (20 points) You have a RLC circuit driven by an AC voltage source, with $V(t) = V_0 \sin\omega t$. However, this is no ordinary circuit, because you've laid your hands on a negative resistance, so that while $L, C > 0$, you have $R < 0$.

(a) Find the general solution for the current, $I(t)$, including undetermined constants.

Answer: The ODE for the current is

$$\frac{dV}{dt} = V_0\omega\cos\omega t = \frac{1}{C}I + R\frac{dI}{dt} + L\frac{d^2I}{dt^2}$$

Dividing out by L and introducing our constants, we get

$$v\cos\omega t = \ddot{I} - 2r\dot{I} + cI$$

First, let's get the homogeneous solution to

$$\left(\frac{d}{dt} - a\right)\left(\frac{d}{dt} - b\right)I_h = 0$$

With

$$a, b = r \pm \sqrt{r^2 - c}$$

The solutions are therefore

$$\begin{aligned} I_h &= e^{rt}\left(Ae^{kt} + Be^{-kt}\right) &\quad \text{when} \quad r^2 \geq c; \quad k = \sqrt{r^2 - c} \\ I_h &= e^{rt}\left(Ae^{i\kappa t} + Be^{-i\kappa t}\right) &\quad \text{when} \quad r^2 < c; \quad \kappa = \sqrt{c - r^2}\end{aligned}$$

The particular solution will have the form

$$I_p = D\cos\omega t + E\sin\omega t$$

Therefore

$$\dot{I}_p = -D\omega\sin\omega t + E\omega\cos\omega t \quad , \quad \ddot{I}_p = -D\omega^2\cos\omega t - E\omega^2\sin\omega t$$

Putting these into the ODE, we get

$$v\cos\omega t = (cD - 2rE\omega - D\omega^2)\cos\omega t + (cE + 2rD\omega - E\omega^2)\sin\omega t$$

Therefore

$$D = \frac{v(c-\omega^2)}{(c-\omega^2)^2 + 4r^2\omega^2} \quad , \quad E = -\frac{2vr\omega}{(c-\omega^2)^2 + 4r^2\omega^2}$$

The general solution is $I = I_p + I_h$, with A and B undetermined constants.

(b) What does $I(t)$ look like as $t \to \infty$? Can you interpret this physically? (*Hint:* A normal resistance dissipates energy. What would a negative resistance do?) What must the undetermined constants be if your current is to remain finite, so that $|I(\infty)| < \infty$?

Answer: As $t \to \infty$, I_h will dominate in the solution, since it includes an exponentially growing e^{rt} factor. The current blows up. This is not surprising, as it is the opposite of what would happen with a positive resistance. A negative resistance feeds energy into the circuit without limit, so things go crazy at large times. To have the current remain finite, the undetermined constants A and B must both be zero.

To make it easier for me to read your results, use the following constants in your solutions:

$$v \equiv \frac{V_0\omega}{L} \qquad r \equiv -\frac{R}{2L} \qquad c \equiv \frac{1}{CL}$$

4. (25 points) The steady-state temperature of an annular plate is given by the solution to $\nabla^2 T = 0$, where $T = T(r,\theta)$—use the Laplacian in polar coordinates.

(a) Find the general solution for T. (*Hint:* For the angular part, don't forget single-valuedness, so that $T(r,\theta) = T(r,\theta + n2\pi)$ for any integer n. The radial equation will *not* be a Bessel equation. Here, be careful with $m = 0$ as a special case: remember that a linear second order ODE has *two* independent solutions.)

Answer: The PDE is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

Separation of variables means $T(r,\theta) = R(r)\Theta(\theta)$. Plug this in the PDE, and multiply everything by $r^2/R\Theta$, getting

$$\frac{r}{R}\frac{dR}{dr} + \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

The terms here depend only on r or only on θ. Treat the angular term in the standard fashion, setting it equal to $-m^2$. We then get, for the angular part:

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0$$

The solutions are

$$\Theta = A\sin m\theta + B\cos m\theta$$

or

$$\Theta_m = A_m \sin(m\theta + \phi_m)$$

Single-valuedness restricts m to integer values; $m = 0, 1, 2, \ldots$

The radial equation now becomes

$$r^2R'' + rR' - m^2R = 0$$

Note that this is *not* a Bessel equation! The solutions are simpler, of the form $R = x^k$. Plugging in $R' = kr^{k-1}$ and $R'' = k(k-1)r^{k-2}$, we get

$$k(k+1) + k - m^2 = 0 \quad \Rightarrow \quad k^2 = m^2 \quad \Rightarrow \quad k = \pm m$$

So the radial solutions are

$$R_m = C_m r^m + D_m r^{-m}$$

But $m = 0$ is an exception, since the above method only gives one solution, a constant. The second solution is to

$$r^2 R_0'' + r R_0' = 0$$

Say $P = R_0'$. Then we get

$$r P' + P = 0$$

The solution to this is $P = D r^{-1}$, and therefore

$$R_0 = C_0 + D_0 \int \frac{dr}{r} = C_0 + D_0 \ln r$$

When $m = 0$, the angular part also becomes $\Theta_0 = A_0 + B_0\theta$. The general solution then is

$$\begin{aligned} T(r,\theta) &= (C_0 + D_0 \ln r)(A_0 + B_0\theta) + \\ &\quad \sum_{m=1}^{\infty} (C_m r^m + D_m r^{-m}) \sin(m\theta + \phi_m) \end{aligned}$$

(b) Find the exact solution $T(r,\theta)$ in the presence of the following boundary conditions. The inside edge of the plate has $r = a$ and is kept at a temperature $T(a,\theta) = T_0$. The outer edge has $r = 2a$ and is kept at a temperature of $T(2a,\theta) = 2T_0$. (*Hint:* The boundary conditions mean there is no dependence on the angle θ. So, which m values will survive from the general solution?)

Answer: Since there is no angular dependence, all terms except those for $m = 0$ must vanish. B_0 must also be zero; absorb A_0 in the other constants. This leaves us with the boundary values at $r = a$ and $r = 2a$:

$$C_0 + D_0 \ln a = T_0$$

$$C_0 + D_0 \ln 2a = 2T_0$$

Solving, we get

$$D_0 = \frac{T_0}{\ln 2} \quad , \quad C_0 = T_0 \left(1 - \frac{\ln a}{\ln 2}\right)$$

Therefore

$$T = T_0\left(1 + \frac{\ln r/a}{\ln 2}\right)$$

5. (15 points) Show that in empty space, you can get electromagnetic waves. Start by writing down Maxwell's equations for empty space, where there are no currents or charges ($\rho = 0$, $\mathbf{j} = 0$). Then show that you can derive wave equations from these empty space Maxwell's equations. That is, you should get

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2\mathbf{B} = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

Find out what c is in terms of ϵ_0 and μ_0.

Hint: To begin, operate on one of Maxwell's equations with either ∇, $\nabla\cdot$, or $\nabla\times$. Then use the "Table of Vector Identities Involving ∇" in the textbook.

Answer: The empty space Maxwell's equation are:

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \mu_0\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}$$

Operate on both sides of the $\nabla\times\mathbf{E}$ equation with $\nabla\times$ and use the identity $\nabla\times\nabla\times\mathbf{V} = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}$. We get

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}$$

Now replace $\nabla\cdot\mathbf{E}$ and $\nabla\times\mathbf{B}$ with what two of the untouched Maxwell's equations say they are:

$$-\nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\mu_0\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}$$

With some minor algebra, we get:

$$\nabla^2\mathbf{E} = \mu_o\epsilon_0\,\frac{\partial^2\mathbf{E}}{\partial t^2}$$

There's no need to reproduce the procedure to get the wave equation for **B**, since it's practically identical. Looking at the wave equations, we notice that the wave speed must be

$$c = \frac{1}{\sqrt{\mu_o \epsilon_0}}$$