Solutions to Midterm 1

1. (30 points) You have a Hermitian matrix

$$M = \begin{pmatrix} 3 & -i & 0 \\ i & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(a) Diagonalize M. That is, find the set of eigenvalues $\{\lambda_n\}$ and *orthonormal* eigenvectors for M, and construct the unitary matrix U such that $\Lambda = U^\dagger M\, U$ is diagonal—the matrix elements $\Lambda_{mn} = \lambda_n \delta_{mn}$.

Hint: If you get degeneracy in the eigenvalues, the eigenvectors corresponding to them won't be automatically orthogonal. (They might be if you are lucky.) So be careful to check orthogonality, and to apply the appropriate fix (subtracting out parallel components) if you need to.

Answer: The usual procedure:

$$\begin{vmatrix} 3-\lambda & -i & 0 \\ i & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(3-\lambda)^2 - 1\right] = 0$$

The solutions are the eigenvalues:

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = 2$$

We can plugging these back in and solve for the eigenvectors. If we choose the first element of one of the degenerate $\lambda = 2$ vectors as 1 and the other one zero, we get, after dividing each vector by its length to normalize it:

$$\alpha_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad \alpha_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These are all orthogonal, so there is no need for any fixing in the degenerate subspace. If you chose a non-zero first element for the third eigenvector, say 1, you would have got

$$\alpha_3' = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}$$

To make sure this was orthogonal to α_2, we would have to subtract out the parallel component to get:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-i\\1\end{pmatrix}-\left[\frac{1}{\sqrt{6}}\begin{pmatrix}1 & i & 0\end{pmatrix}\begin{pmatrix}1\\-i\\1\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

And if you normalize this again, you just get α_3.

The diagonalizing matrix is then

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\\ i & -i & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}$$

(b) Say you have an arbitrary matrix A. Say also that U_A diagonalizes A, so that $\Lambda_A = U^\dagger A\, U$ is diagonal, and $\Lambda_{A\,mn} = \lambda_{A\,n}\delta_{mn}$, with $\{\lambda_{A\,n}\}$ the eigenvalues of A. An arbitrary function of A is defined as

$$f(A)=\sum_{j=0}^{\infty} f_j A^j$$

The f_n are real numbers. Now prove that

(i) U_A also diagonalizes $f(A)$; in other words, $\Lambda_f = U^\dagger f(A)\, U$ is a diagonal matrix.

(ii) The matrix elements of Λ_f are $\Lambda_{f\,mn} = f(\lambda_{A\,n})\delta_{mn}$; in other words, that the eigenvalues of $f(A)$ are just $f(\lambda_{A\,n})$, the function applied to the eigenvalues of A.

Hint: Remember that $U_A^\dagger = U_A^{-1}$. Start by showing that $(U_A^\dagger A\, U_A)^n = U_A^\dagger A^n U_A$.

Answer: Using the hint:

$$\begin{aligned}(U_A^\dagger A\, U_A)^n &= U_A^\dagger A\, U_A U_A^\dagger A\, U_A U_A^\dagger A\, U_A \cdots U_A^\dagger A\, U_A\\ &= U_A^\dagger A\,(U_A U_A^\dagger) A\,(U_A U_A^\dagger) A\, U_A \cdots U_A^\dagger A\, U_A\\ &= U_A^\dagger A\, A\, A \cdots A\, U_A\\ &= U_A^\dagger A^n U_A\end{aligned}$$

Now apply this to diagonalizing $f(A)$:

$$\Lambda_f = U^\dagger f(A)\, U = \sum_{j=0}^{\infty} f_j U^\dagger A^j\, U = \sum_{j=0}^{\infty} f_j (U^\dagger A\, U)^j = \sum_{j=0}^{\infty} f_j \Lambda_A^j$$

Since Λ_A is a diagonal matrix, any power Λ_A^j is also diagonal. The sum of diagonal matrices is also diagonal—therefore Λ_f is diagonal. That takes care of **(i)**.

For **(ii)**, we need to know that for a diagonal matrix Λ_A, the power

$$\Lambda^2_{A\,ik} = \sum_j A_{ij} A_{jk} = \sum_j \lambda_{A\,i} \lambda_{A\,k}\, \delta_{ij} \delta_{jk} = \lambda^2_{A\,i}\, \delta_{ik}$$

This easily generalizes to any integer power of Λ_A. Therefore

$$\Lambda_{f\,mn} = \sum_j f_j \Lambda^j_{A\,mn} = \sum_j f_j \lambda^j_{A\,n} \delta_{mn} = f(\lambda_{A\,n}) \delta_{mn}$$

(c) Find e^{aM}, where a is an arbitrary constant.

Hint: You may find it difficult to do this by the usual way of finding M^2, M^3 etc. and seeking a pattern. Use the results of parts (a) and (b) instead: find $e^{a\Lambda}$, and then use $U\, U^\dagger = I$ and therefore

$$e^{aM} = U\, U^\dagger\, e^{aM}\, U\, U^\dagger = U\, e^{a\Lambda}\, U^\dagger$$

Answer: The diagonalized matrix from (a) was:

$$\Lambda = U^\dagger M\, U = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Now get the exponential,

$$e^{a\Lambda} = \begin{pmatrix} e^{4a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix}$$

Therefore

$$
\begin{aligned}
e^{aM} &= U\, e^{a\Lambda}\, U^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} e^{4a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix}\begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{4a}+e^{2a} & -i(e^{4a}-e^{2a}) & 0 \\ i(e^{4a}-e^{2a}) & e^{4a}+e^{2a} & 0 \\ 0 & 0 & 2e^{2a} \end{pmatrix} \\
&= \begin{pmatrix} e^{3a}\cosh a & -i\, e^{3a}\sinh a & 0 \\ i\, e^{3a}\sinh a & e^{3a}\cosh a & 0 \\ 0 & 0 & e^{2a} \end{pmatrix}
\end{aligned}
$$

2. (20 points) Find the interval of convergence of this series for a function of x:

$$f(x) = \sum_{n=1}^{\infty} \frac{n+\ln n}{n^2+1}\, e^{-nx}$$

Answer: Start with the ratio test. Convergence happens for:

$$\rho = \lim_{n\to\infty}\left|\frac{n+1+\ln(n+1)}{(n+1)^2+1}\, e^{-(n+1)x}\, \frac{n^2+1}{n+\ln n}\, e^{nx}\right| = \left|e^{-x}\right| < 1$$

Since e^{-x} is always positive, this translates to $0 < x < \infty$.

Now we check the limits, or rather $x = 0$ only, since that is the only finite limit. This calls for the special comparison test with the divergent series $\sum \frac{1}{n}$, since as $n \to \infty$,

$$\frac{n+\ln n}{n^2+1} \to \frac{1}{n}$$

Since

$$\lim_{n\to\infty} \frac{n+\ln n}{n^2+1}\, n = 1 \neq 0$$

the series must diverge at $x = 0$. The interval of convergence is $0 < x < \infty$.

3. (20 points) Find (x, y) that maximizes

$$f(x, y) = y^3 + xy^2$$

subject to the constraint that

$$2y^2 + xy = 1$$

Answer: Use Lagrange multipliers, setting up

$$F(x,y) = y^3 + xy^2 - \lambda\left(2y^2 + xy - 1\right)$$

taking the derivatives,

$$\frac{\partial F}{\partial x} = y^2 - \lambda y = 0 \qquad \Rightarrow \qquad \lambda = y$$

$$\frac{\partial F}{\partial y} = 3y^2 + 2xy - \lambda(4y + x) = 0 \qquad \Rightarrow \qquad y = x$$

$$\frac{\partial F}{\partial \lambda} = 2y^2 + xy - 1 = 0 \qquad \Rightarrow \qquad 3y^2 = 1$$

So we get

$$x = y = \frac{1}{\sqrt{3}}$$

4. (30 points) You have a function

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ +1 & 0 < x < a \\ 0 & a < |x| < L \end{cases}$$

where $L > a > 0$ and $f(x)$ is periodic with period $2L$.

(a) Find the complex Fourier coefficients c_n in the Fourier expansion of $f(x)$.

Answer: Do the integral for the c_n. For $n \neq 0$,

$$\begin{aligned} c_n &= -\frac{1}{2L}\left(\int_{-a}^{0} dx\, e^{-in\pi x/L} + \int_0^a dx\, e^{-in\pi x/L}\right) \\ &= -\frac{1}{2in\pi}\left[- e^{-in\pi x/L}\Big|_{-a}^{0} + e^{-in\pi x/L}\Big|_0^a\right] = \frac{i}{n\pi}\left[\cos\left(\frac{n\pi a}{L}\right) - 1\right] \end{aligned}$$

For $n = 0$, the integral gives $c_0 = 0$.

(b) Find the Fourier transform $\tilde{f}(k)$ of the nonperiodic function

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ +1 & 0 < x < a \\ 0 & a < |x| < \infty \end{cases}$$

Answer:

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \left(-\int_{-a}^{0} dx\, e^{-ikx} + \int_{0}^{a} dx\, e^{-ikx} \right) \\ &= -\frac{1}{\sqrt{2\pi}} \frac{1}{ik} \left(-\, e^{-ikx}\Big|_{-a}^{0} + e^{-ikx}\Big|_{0}^{a} \right) = \sqrt{\frac{2}{\pi}} \frac{i}{k} \left(\cos ka - 1\right) \end{aligned}$$

(c) Do your answers to (a) and (b) fit what you'd expect for an even or odd function (real, imaginary, zero, nonzero sort of thing)? Does Lc_n compare to $\tilde{f}(k)$ as it should, when $L \to \infty$—what is the relationship between k and n in this limit?

Answer: Yes. $f(x)$ is an odd function, so c_n should be imaginary, which they are. The same goes for $\tilde{f}(k)$: it's imaginary. And as $L \to \infty$,

$$Lc_n \to \frac{i}{n\pi/L} \left[\cos\left(\frac{n\pi}{L} a \right) - 1 \right] = \frac{i}{k} (\cos ka - 1)$$

where $k = n\pi/L$, as it should be, when you look at $\tilde{f}(k)$.