Homework Solutions # 1 (Liboff Chapter 3)

3.2

- No inverse of $\hat{D}$. The integral $\int^x dx' \partial\phi/\partial x' = \phi + c$; only up to an arbitrary additive constant. No operator which destroys information can have an inverse.
- $\hat{I}^{-1} = \hat{I}$.
- $\hat{F}^{-1} =$ multiplication by $1/F(x)$, except where $F(x) = 0$.
- $\hat{B}^{-1} =$ multiplication by 3.
- $\hat{\Theta}$ has no inverse.
- $\hat{G}$ has no inverse.

3.4 Plug ϕ_β into the eigenvalue equation, getting

$$e^{\beta(x+\zeta)} g(x+\zeta) = a_\beta e^{\beta x} g(x)$$

where a_β is the eigenvalue. Using $g(x+\zeta) = g(x)$, we get $a_\beta = e^{\beta\zeta}$.

3.11

$$\frac{\partial\psi}{\partial x} = \left(\frac{ip_0}{\hbar} - \frac{x - x_0}{2a^2}\right)\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{1}{2a^2}\psi + \left(\frac{ip_0}{\hbar} - \frac{x - x_0}{2a^2}\right)^2 \psi$$

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} dx\, \psi^* \frac{\partial^2\psi}{\partial x^2}$$

Using, as the book does, $\eta = (x - x_0)/a$,

$$\langle p^2 \rangle = -\hbar^2 a |A|^2 \int_{-\infty}^{\infty} d\eta\, e^{-\eta^2/2} \left(-\frac{1}{2a^2} - \frac{p_0^2}{\hbar^2} - \frac{ip_0\eta}{a\hbar} + \frac{\eta^2}{4a^2}\right)$$

The $i\eta$ term will integrate to zero, as an odd function. The constant coefficient terms can be evaluated directly, using $\int_{-\infty}^{\infty} d\eta\, e^{-\eta^2/2} = \sqrt{2\pi}$. But we also need to do

$$\int_{-\infty}^{\infty} d\eta\, \eta^2 e^{-\eta^2/2} = -2\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty} d\eta\, e^{-\alpha\eta^2/2}\bigg|_{\alpha=1} = -2\frac{\partial}{\partial\alpha}\sqrt{2\pi/\alpha}\bigg|_{\alpha=1} = \sqrt{2\pi}$$

So, putting all this together,

$$\langle p^2\rangle = \frac{\hbar^2}{4a^2} + p_0^2$$

In the end,

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar^2}{4a^2} + p_0^2 - p_0^2} = \frac{\hbar}{2a}$$

The uncertainty relation holds:

$$\Delta p \Delta x = \frac{\hbar}{2}$$

3.15 ψ is an odd function of x, and therefore $\partial\psi/\partial x$ is an even function. So $-i\hbar\int dx\, \psi^*\partial\psi/\partial x$ is an integral of an odd function, which results in 0.

3.16

$$\sum_{l=0}^{\infty} b_l \hat{A}^l \phi_n = \sum_{l=0}^{\infty} b_l a_n^l \phi_n = f(a_n)\phi_n$$

3.19 Using $\sin^2\theta + \cos^2\theta = 1$ works here as well, so $\hat{O}_1 = \partial^2/\partial x^2$. Then, use $\cos 2\theta = 1 - 2\sin^2\theta$ to get $\hat{O}_2 = 1 + \int_a^b dx$.

3.20 Plug ψ into the equation, using $a\partial\psi_1/\partial t = \hat{H}_1\psi$ and $1 \to 2$.

$$a\left(\frac{\partial\psi_1}{\partial t}\psi_2 + \psi_1\frac{\partial\psi_2}{\partial t}\right) = \psi_2\hat{H}_1\psi_1 + \psi_1\hat{H}_2\psi_2 = a\left(\frac{\partial\psi_1}{\partial t}\psi_2 + \psi_1\frac{\partial\psi_2}{\partial t}\right)$$

This obviously wouldn't work if you had to do a second derivative, as you'd pick up the extra mixed term,

$$2a^2\frac{\partial\psi_1}{\partial t}\frac{\partial\psi_2}{\partial t}$$

which you can't obtain from individual $a^2 \partial\psi_1/\partial t = \hat{H}_1\psi$ solutions.

3.23 We start with an energy eigenstate of the free particle, and apply the boundary condition $\phi(0) = 0$. This means, if $\phi(x) = Ae^{ikx} + Be^{-ikx}$, then $A+B = 0$. In other words, $\phi_k(x) = a \sin kx$ where a is an arbitrary constant; $E_k = \hbar^2 k^2/2m$. For $x \geq 0$, $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$. So

$$\hat{H}\phi_k(x) = \frac{\hbar^2 k^2}{2m} a \sin kx = E_k \phi_k(x)$$

The time dependent state is

$$\psi_k(x,t) = a \sin kx \, e^{-iE_k t/\hbar}$$