Homework Solutions # 10 (Liboff Chapter 11)

11.6

(a) Write out the matrix multiplication result:

$$\begin{aligned}\hat{R}(\phi_1)\hat{R}(\phi_2) &= \begin{pmatrix} \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 & \cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2 \\ -\sin\phi_1\cos\phi_2 - \cos\phi_1\sin\phi_2 & -\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi_1+\phi_2) & \sin(\phi_1+\phi_2) \\ -\sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) \end{pmatrix} = \hat{R}(\phi_1+\phi_2)\end{aligned}$$

(b) Since $\hat{R}(\phi_1)\hat{R}(\phi_2) = \hat{R}(\phi_1+\phi_2) = \hat{R}(\phi_2)\hat{R}(\phi_1)$, then $[\hat{R}(\phi_1), \hat{R}(\phi_2)] = 0$.

(c) We can see that $\tilde{R}(\phi) = \hat{R}(-\phi) = \hat{R}^{-1}(\phi)$.

(d) Solve for the eigenvalues a from

$$\begin{vmatrix} \cos\phi - a & \sin\phi & 0 \\ -\sin\phi & \cos\phi - a & 0 \\ 0 & 0 & 1-a \end{vmatrix} = (1-a)[(\cos\phi - a)^2 + \sin^2\phi] = 0$$

After some algebra, the solutions are $a = 1$, $a = \cos\phi \pm i\sin\phi = e^{\pm i\phi}$. For all three, $|a|^2 = 1$.

11.45 Just plug the various α's and β's into the eigenvalue equations. You'll find $\frac{\hbar}{2}\sigma_x\,\alpha_x = \frac{\hbar}{2}\alpha_x$, $\frac{\hbar}{2}\sigma_x\,\beta_x = -\frac{\hbar}{2}\beta_x$, $\frac{\hbar}{2}\sigma_y\,\alpha_y = \frac{\hbar}{2}\alpha_y$, $\frac{\hbar}{2}\sigma_y\,\beta_y = -\frac{\hbar}{2}\beta_y$.

Keeping in mind that for a vector v, $v^\dagger = \tilde{v}^*$, you can verify that $\alpha_x^\dagger\alpha_x = \beta_x^\dagger\beta_x = 1$, $\alpha_x^\dagger\beta_x = 0$, and similar relations with $x \to y$. So they're orthonormal.

11.47

(a) $|\uparrow\rangle$, or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b)

$$\langle S_x^2\rangle = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\langle S_z^2 \rangle = -\frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\frac{1}{2}\langle S^2 - S_z^2 \rangle = -\frac{\hbar^2}{8} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$$

11.51 One possible wavefunction is

$$\varphi = A \, \frac{e^{ikx}}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where A depends on the current density. Since $E = \hbar^2 k^2 / 2m_e$,

$$k = \frac{\sqrt{2m_e E}}{\hbar} = 1.98 \times 10^{11} \text{ m}^{-1}$$

$$\omega = \frac{E}{\hbar} = 2.28 \times 10^{18} \text{ s}^{-1}$$