Homework Solutions # 2 (Liboff Chapter 4)

4.5

(a) Given in book.

(b) $|\varphi\rangle\langle\psi|f\rangle = a|\varphi\rangle$, where $a = \int dx\, \psi^* f$.

(c) $\langle f|\varphi\rangle\langle\psi|f\rangle$.

(d) $\langle f|\varphi\rangle\langle\psi|\psi\rangle = \langle f|\varphi\rangle$.

4.8

$$g(x) = \sqrt{\frac{2}{a}} \sum_n a_n \sin\frac{n\pi x}{a}$$

where

$$a_n = \sqrt{\frac{2}{a}} \int_0^a dx\, \sin\frac{n\pi x}{a} x(x-a)e^{ikx} = \frac{1}{2i}\sqrt{\frac{2}{a}} \int_0^a dx\, (x^2-ax)\left(e^{i(k+\frac{n\pi}{a})x} - e^{-i(k+\frac{n\pi}{a})x}\right)$$

Now the tedious integration. Use

$$I(\alpha) = \int_0^a dx\, e^{i\alpha x} = \frac{1}{i\alpha}\left(e^{ia\alpha} - 1\right)$$

$$\int_0^a dx\, x\, e^{i\alpha x} = -i\frac{\partial I}{\partial\alpha} = \frac{1}{\alpha^2}\left(e^{ia\alpha} - 1\right) - \frac{ia}{\alpha}e^{ia\alpha}$$

$$\int_0^a dx\, x^2\, e^{i\alpha x} = -\frac{\partial^2 I}{\partial\alpha^2} = \frac{2i}{\alpha^3}\left(e^{ia\alpha} - 1\right) + \frac{2a}{\alpha^2}e^{ia\alpha} - \frac{ia^2}{\alpha}e^{ia\alpha}$$

Afer a bunch of algebra I'm not going to reproduce here, we get

$$a_n = \sqrt{\frac{2}{a}}\left\{\left[(-1)^n e^{ika} - 1\right]\left[\frac{1}{(k+n\pi/a)^3)} - \frac{1}{(k+n\pi/a)^3)}\right]\right.$$

$$-ia(-1)^n e^{ika}\left[\frac{1}{(k+n\pi/a)^2} - \frac{1}{(k-n\pi/a)^2}\right]$$

$$\left.-\frac{a}{2i}\left[(-1)^n e^{ika} - 1\right]\left[\frac{1}{(k+n\pi/a)^2} - \frac{1}{(k-n\pi/a)^2}\right]\right\}$$

4.9

$\|\psi+\varphi\|^2 = \langle\psi+\varphi|\psi+\varphi\rangle = (\langle\psi|+\langle\varphi|)(|\psi\rangle+|\varphi\rangle) = \langle\psi|\psi\rangle+\langle\varphi|\psi\rangle+\langle\psi|\varphi\rangle+\langle\varphi|\varphi\rangle$

Orthogonality means $\langle\varphi|\psi\rangle = 0$, so

$$\|\psi + \varphi\|^2 = \langle\psi|\psi\rangle + \langle\varphi|\varphi\rangle = \|\psi\|^2 + \|\varphi\|^2$$

4.12

(a) $\left[i(\hat{A}\hat{B} - \hat{B}\hat{A})\right]^\dagger = -i(\hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger) = i(\hat{A}\hat{B} - \hat{B}\hat{A})$, so Hermitian.

(b) Same, except for no factor of i, so it's anti-Hermitian ($\hat{O}^\dagger = -\hat{O}$).

(c) $\frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})^\dagger = \frac{1}{2}(\hat{B}^\dagger\hat{A}^\dagger + \hat{A}^\dagger\hat{B}^\dagger) = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$, so Hermitian.

(d) $(\hat{A}^\dagger\hat{A})^\dagger = \hat{A}^\dagger\hat{A}^{\dagger\dagger} = \hat{A}^\dagger\hat{A}$, so Hermitian.

(e) $\frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$.

4.14 First, by the properties of the inner product,

$$\langle\varphi|\hat{A}\varphi\rangle = \langle\hat{A}\varphi|\varphi\rangle^*$$

Then, by the definition of the Hermitian adjoint, and the hermiticity of $\hat{A}$,

$$\langle\varphi|\hat{A}\varphi\rangle = \langle\hat{A}^\dagger\varphi|\varphi\rangle = \langle\hat{A}\varphi|\varphi\rangle$$

So, since $\langle\hat{A}\varphi|\varphi\rangle^* = \langle A\rangle^*$, $\langle A\rangle = \langle A\rangle^*$.

4.16 If $\hat{B}|\varphi\rangle = b_1|\varphi\rangle$, then $\langle\varphi|\hat{B}\varphi\rangle = b_1\langle\varphi|\varphi\rangle = b_1$. But $\langle\hat{B}\varphi|\varphi\rangle = \langle b_1\varphi|\varphi\rangle = b_1^*\langle\varphi|\varphi\rangle = b_1^* \neq b_1$. So $\hat{B}^\dagger \neq \hat{B}$.

4.35

(a) Normalize by requiring $\int|\psi|^2 = A^2\int_0^a dx = A^2 a = 1$. So $A = 1/\sqrt{a}$.

(b) Note that ψ is antisymmetric about $x = a/2$. This means $\langle n|\psi\rangle = \sqrt{2/a}\int\psi\sin(n\pi x/a) = 0$ for odd n, because those basis functions are symmetric about $x = a/2$. So the first non-zero energy component of ψ is $n = 2$.

4.36

(a)

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\psi$$

So it's an energy eigenstate.

(b) No; it's a sum of $p = \pm\hbar k$ states.

(c) $p = \pm\hbar k$ can be found; in terms of momentum eigenstates

$$|\psi\rangle = A| + \hbar k\rangle + \frac{A}{\sqrt{2}}| - \hbar k\rangle$$

So

$$P(+\hbar k) = |\langle +\hbar k|\psi\rangle|^2 = A^2$$

$$P(-\hbar k) = |\langle -\hbar k|\psi\rangle|^2 = A^2/2$$

Since $P(+) + P(-) = 1$, $A^2 = \frac{2}{3}$, and so

$$P(+\hbar k) = \frac{2}{3} \qquad P(-\hbar k) = \frac{1}{3}$$