Homework Solutions # 3 (Liboff Chapter 5)

5.2

(a) If we interpret ψ to be a probability amplitude for each neutron, $\int dx|\psi|^2 = 1$. We get

$$A^2 \int_0^a dx\,(x-a)^2 x^2 = A^2 \left[\frac{x^5}{5} - \frac{ax^4}{2} + \frac{a^2x^3}{3}\right]_0^a = A^2 \frac{a^5}{30} = 1$$

So $A = \sqrt{30/a^5}$. If we interpret $|\psi|^2$ to be the neutron density, we should multiply A by $\sqrt{1000}$. Let's stick with the first.

(b) Since ψ is symmetric about $x = a/2$, half the neutrons must be between 0 and $a/2$. 500.

(c) Find $\langle 5|\psi\rangle$:

$$\sqrt{\frac{2}{a}}\, A \int_0^a dx\, x(x-a) \sin\frac{5\pi x}{a} =$$

$$\sqrt{\frac{60}{a^6}} \left[\frac{2x}{(5\pi/a)^2} \sin\frac{5\pi x}{a} + \left(\frac{2}{(5\pi/a)^3} - \frac{x^2}{5\pi/a}\right) \cos\frac{5\pi x}{a}\right.$$

$$\left. - \frac{a}{(5\pi/a)^2} \sin\frac{5\pi x}{a} + \frac{ax}{5\pi/a} \cos\frac{5\pi x}{a}\right]_0^a = -\frac{8\sqrt{15}}{(5\pi)^3}$$

For the probability,

$$P(5) = |\langle 5|\psi\rangle|^2 = 6.4 \times 10^{-5}$$

For 1000 particles, this still means 6.4×10^{-2}.

(d) Using $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$,

$$\langle E\rangle = -\frac{\hbar^2 A^2}{2m} \int_0^a dx\,(x^2 - ax) \frac{\partial^2}{\partial x^2}(x^2 - ax) = -\frac{30\hbar^2}{ma^5}\left(\frac{a^3}{3} - \frac{a^3}{2}\right) = \frac{5\hbar^2}{ma^2}$$

Note that this is just barely above $E_1 = \hbar^2\pi^2/2ma^2$.

5.11 We start with the wavefunction $\psi = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$, but with the wall now moved to $x = 2a$, and the new energy eigenstates $\phi_n = \sqrt{\frac{1}{a}} \sin \frac{n\pi x}{2a}$. Now,

$$\langle \psi | n \rangle = \frac{\sqrt{2}}{a} \int_0^a dx \, \sin \frac{\pi x}{a} \, \sin \frac{n \pi x}{2a}$$

From symmetry, this is zero for even $n > 2$. $|\langle \psi | 2 \rangle|^2 = \frac{1}{2}$, so E_2 is the most likely energy, which is $\hbar^2 \pi^2 / 2ma^2$, the same as the initial energy.

5.18

$$[\hat{A}, \hat{B}] |\psi\rangle = \sum_n (\hat{A}\hat{B} - \hat{B}\hat{A}) |n\rangle \langle n | \psi \rangle = \sum_n (a_n b_n - b_n a_n) |n\rangle \langle n | \psi \rangle = 0$$

5.21 Expand the operators in the basis set of the eigenvectors of $\hat{A}$:

$$\hat{A} = \sum_n \hat{A} |n\rangle \langle n| = \sum_n a_n |n\rangle \langle n|$$

$$\hat{B} = \sum_{mn} |n\rangle \langle n | \hat{B} | m \rangle \langle m| = \sum_{mn} B_{nm} |n\rangle \langle m|$$

where a_n are the eigenvalues of $\hat{A}$ and $B_{nm} = \langle n | \hat{B} | m \rangle$ (the "matrix elements" of $\hat{B}$). Note that in the basis of its own eigenvectors, an operator such as $\hat{A}$ is "diagonal," in that its off-diagonal ($m \neq n$) matrix elements are all 0.

Then, noting that $\langle k | n \rangle = \delta_{kn}$,

$$\hat{A}\hat{B} = \sum_{kmn} a_k B_{nm} |k\rangle \langle k | n \rangle \langle m| = \sum_{mn} a_n B_{nm} |n\rangle \langle m|$$

$$\hat{B}\hat{A} = \sum_{kmn} a_k B_{nm} |n\rangle \langle m | k \rangle \langle k| = \sum_{mn} a_m B_{nm} |n\rangle \langle m|$$

So

$$[\hat{A}, \hat{B}] = \sum_{mn} (a_n - a_m) B_{nm} |n\rangle \langle m|$$

This will only vanish if the matrix element $(a_n - a_m) B_{nm}$ is zero for all n and m.

Now, we know that $\hat{A}$ and $\hat{B}$ share one eigenvector, say $|s\rangle$. This means that the matrix element

$$B_{ns} = \langle n | \hat{B} | s \rangle = B_{ss} \delta_{ns}$$

if $\hat{B}^\dagger = \hat{B}$, then also $B_{sn} = B_{ss}\delta_{ns}$. But this is not enough to conclude that $(a_n - a_m)B_{nm}$ vanishes for n and m not equal to s. Therefore, we can't say that $[\hat{A}, \hat{B}] = 0$ in general; they commute only in the subspace spanned by common eigenvectors.

If all eigenstates of $\hat{A}$ are degenerate, however, then $(a_n - a_m)B_{nm} = 0$, because all $a_n = a_m$. Then we can say the operators commute.

5.25 First, we can show that

$$[\hat{x}, \hat{p}^n] = in\hbar\hat{p}^{n-1}$$

$$[\hat{x}^n, \hat{p}] = in\hbar\hat{x}^{n-1}$$

Start with

$$\begin{aligned} [\hat{x}^n, \hat{p}] &= \hat{x}^n\hat{p} - \hat{p}\hat{x}^n = \hat{x}^{n-1}\hat{p}\hat{x} + \hat{x}^{n-1}[\hat{x}, \hat{p}] - \hat{p}\hat{x}^n \\ &= (\hat{x}^{n-1}\hat{p} - \hat{p}\hat{x}^{n-1})\hat{x} + i\hbar\hat{x}^{n-1} = [\hat{x}^{n-1}, \hat{p}]\hat{x} + i\hbar\hat{x}^{n-1} \end{aligned}$$

Then, by induction, you can show both relations.

Now, using $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$, $\hat{T} = \frac{1}{2m}\hat{p}^2$, and $V(\hat{x}) = \sum_n v_n\hat{x}^n$:

(a) Noting that $[\hat{x}, V(\hat{x})] = 0$,

$$[\hat{x}, \hat{H}] = \left[\hat{x}, \frac{1}{2m}\hat{p}^2 + V(\hat{x})\right] = \frac{1}{2m}[\hat{x}, \hat{p}^2] = \frac{i\hbar\hat{p}}{m}$$

Therefore

$$\Delta x\Delta E \geq \frac{1}{2}|\langle i\hbar p/m\rangle| = \frac{\hbar}{2m}|\langle p\rangle|$$

(b) Now,

$$\begin{aligned} [\hat{p}, \hat{H}] &= [\hat{p}, V(\hat{x})] = \left[\hat{p}, \sum_n v_n\hat{x}^n\right] = \sum_n v_n[\hat{p}, \hat{x}^n] = i\hbar\sum_n nv_n\hat{x}^{n-1} \\ &= i\hbar\frac{\partial V}{\partial x}(\hat{x}) \end{aligned}$$

If $-\frac{\partial V}{\partial x} = F$, the force,

$$\Delta p\Delta E \geq \frac{\hbar}{2}|\langle F\rangle|$$

(c) $[\hat{x}, \hat{T}] = [\hat{x}, \hat{H}]$, so the uncertainty is the same.

(d) $[\hat{p}, \hat{T}] = 0$, so the uncertainty is also zero.

5.28

$$[\hat{t}, \hat{E}] = i\hbar \left[t\frac{\partial}{\partial t} - \frac{\partial}{\partial t} \cdot t \right] = i\hbar \left[t\frac{\partial}{\partial t} - 1 - t\frac{\partial}{\partial t} \right] = -i\hbar$$

So

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

5.50 Since $[\hat{p}, \sin \hat{p}] = 0$,

$$[\hat{p}, \hat{x} \sin \hat{p}] = \hat{p}\hat{x} \sin \hat{p} - \hat{x}\hat{p} \sin \hat{p} = [\hat{p}, \hat{x}] \sin \hat{p} = -i\hbar \sin \hat{p}$$

5.51

(a)

$$\psi(x) = \frac{b_0}{\sqrt{2\pi}} \int_{-\pi/a}^{\pi/a} dk \, e^{ikx} = \frac{b_0}{\sqrt{2\pi}} \left. \frac{e^{ikx}}{ix} \right|_{-\pi/a}^{\pi/a} = \frac{b_0}{x} \sqrt{\frac{2}{\pi}} \sin \frac{\pi x}{a}$$

Normalization:

$$\frac{2}{\pi} b_0^2 \int_{-\infty}^{\infty} dx \, \frac{1}{x^2} \, \sin^2 \frac{\pi x}{a} = \frac{2}{\pi} b_0^2 \frac{\pi^2}{a} = 1 \Rightarrow b_0 = \sqrt{\frac{a}{2\pi}}$$

Since $b(k)^* b(k) \, dk$ must be dimensionless (probability), b_0 must have the dimensions of a square root of a length. It does.

(b) The particle won't be found where $\psi(x) = 0$, which is when $x = na$, $n = \pm 1, \pm 2, \ldots$.