
Homework Solutions # 4 (Liboff Chapter 6)

6.7 Note that $\psi(x, 0)$ is an eigenstate of \hat{H} .

(a)

$$\psi(x, t) = \psi(x, 0)e^{-i\omega t}$$

where $\omega = E/\hbar = \hbar k_0^2/2m$.

(b)

$$\psi(x, t) = \frac{1}{2i} (e^{ik_0x} - e^{-ik_0x}) e^{-i\omega t}$$

in terms of \hat{p} eigenfunctions. Therefore, $\pm\hbar k_0$ can be observed, with equal probability ($\frac{1}{2}$).

(c) at $t = 3$ s, $\psi = e^{ik_0x}$. So

$$\psi(x, t) = e^{ik_0x} e^{-i\omega(t-3)}$$

(The 3 s part is irrelevant; just an overall phase.)

6.13 At $t = 0$, the state is $|\psi_0\rangle$, with $\hat{A}|\psi_0\rangle = a|\psi_0\rangle$. For $t > 0$, using the time evolution operator $\hat{U} = \exp(-\frac{it}{\hbar}\hat{H})$,

$$|\psi\rangle = e^{-\frac{it}{\hbar}\hat{H}}|\psi_0\rangle$$

Then, since $\hat{U}^\dagger = \exp(\frac{it}{\hbar}\hat{H}) = \hat{U}^{-1}$,

$$\langle A \rangle = \langle \psi_0 | e^{\frac{it}{\hbar}\hat{H}} \hat{A} e^{-\frac{it}{\hbar}\hat{H}} | \psi_0 \rangle$$

We have to move \hat{A} to the other side of \hat{U} , so we need the commutator $[\hat{U}, \hat{A}]$. Start with using $\hat{H}\hat{A} = \hat{A}\hat{H} + c$, and finding

$$[\hat{H}^n, \hat{A}] = \hat{H}^n \hat{A} - \hat{A} \hat{H}^n = \hat{H}^{n-1} \hat{A} \hat{H} + c \hat{H}^{n-1} - \hat{A} \hat{H}^n = \dots = nc \hat{H}^{n-1}$$

after interchanging \hat{A} and \hat{H} n times. So

$$\begin{aligned} \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-it}{\hbar} \right)^n \hat{H}^n, \hat{A} \right] &= \sum_n \frac{nc}{n!} \left(\frac{-it}{\hbar} \right)^n \hat{H}^{n-1} = -\frac{ict}{\hbar} \sum_m \frac{1}{m!} \left(\frac{-it}{\hbar} \right)^m \hat{H}^m \\ &= -\frac{ict}{\hbar} e^{-\frac{it}{\hbar}\hat{H}} = -\frac{ict}{\hbar} \hat{U} \end{aligned}$$

So, going back,

$$\langle A \rangle = \langle \psi_0 | \hat{U}^{-1} \hat{U} \hat{A} | \psi_0 \rangle - \langle \psi_0 | \hat{U}^{-1} \hat{U} \left(-\frac{ict}{\hbar} \right) | \psi_0 \rangle = a + \frac{ic}{\hbar} t$$

6.18

(a)

$$f_+(-x) = \frac{f(-x) + f(x)}{2} = f_+(x)$$

$$f_-(-x) = \frac{f(-x) - f(x)}{2} = -f_-(x)$$

(b)

$$\hat{\mathcal{P}}_+ f(x) = \frac{1 + \hat{\mathcal{P}}}{2} f(x) = \frac{f(x) + f(-x)}{2} = f_+(x)$$

(c)

$$\hat{\mathcal{P}}_- f(x) = \frac{1 - \hat{\mathcal{P}}}{2} f(x) = \frac{f(x) - f(-x)}{2} = f_-(x)$$

(d)

$$\hat{\mathcal{P}}_{\pm}^2 = \left(\frac{1 \pm \hat{\mathcal{P}}}{2} \right)^2 = \frac{1 \pm 2\hat{\mathcal{P}} + \hat{\mathcal{P}}^2}{4} = \frac{2 \pm 2\hat{\mathcal{P}}}{4} = \frac{1 \pm \hat{\mathcal{P}}}{2} = \hat{\mathcal{P}}_{\pm}$$

Since $\hat{\mathcal{P}}_{\pm}$ are both linear combinations of 1 and $\hat{\mathcal{P}}$, which commute, $[\hat{\mathcal{P}}_+, \hat{\mathcal{P}}_-] = 0$. And $\hat{\mathcal{P}}_+ + \hat{\mathcal{P}}_- = 1$ trivially, by their definitions.

6.20 States 1 and 3 have even parity, 2 has odd parity. So if -1 is measured for the parity, the state must then be 2; the energy will be measured to be $E_2 = 4E_1$. If $+1$ was found instead, then the even parity subspace will be picked out instead: a combination of state 1 and state 3. The *relative* probabilities will be unaffected by the parity measurement, so the energy will be found to be E_1 with probability $P_1/(P_1 + P_3) = 1/2$, and $E_3 = 9E_1$ with the same probability.

6.21 Note that

$$[\hat{\mathcal{P}}, \hat{x}]f(x) = -xf(-x) - xf(-x)$$

So $[\hat{\mathcal{P}}, \hat{x}] = -2\hat{x}\hat{\mathcal{P}} \neq 0$. Next, check

$$[\hat{\mathcal{P}}, \hat{H}]f(x) = -\frac{\hbar^2}{2m}(f''(-x) - f''(-x)) = 0$$

Since with a change of variable to $u = -x$, differentiating twice gives an overall minus twice. So $[\hat{\mathcal{P}}, \hat{H}] = 0$. Next,

$$[\hat{\mathcal{P}}, \hat{p}]f(x) = -i\hbar(f'(-x) + f'(-x))$$

So $[\hat{\mathcal{P}}, \hat{p}] \neq 0$.

We can therefore get the following sets of commuting observables: $\{\hat{\mathcal{P}}, \hat{H}\}$ or $\{\hat{p}, \hat{H}\}$.

6.23

(a) $\hat{\mathcal{P}}\psi(r, \theta, \phi) = \psi(r, \pi - \theta, \phi + \pi)$

(b) $\hat{\mathcal{P}}\psi(\rho, z, \phi) = \psi(\rho, -z, \phi + \pi)$

(c) $\psi_1(-x, -y, -z) = -\psi_1(x, y, z)$, so its parity is -1 .

$\psi_2((r, \pi - \theta, \phi + \pi) = -\psi_2(r, \theta, \phi)$, since $\cos(\pi - \theta) = -\cos \theta$, so its parity is -1 also.

$\psi_3(\rho, -z, \phi + \pi) = \psi_3(\rho, z, \phi)$, so its parity is $+1$.