Homework Solutions # 5 (Liboff Chapter 7)

7.5 Using the definition of $\hat{a}$ in terms of $\hat{x}$ and $\hat{p}$,

$$[\hat{a}, \hat{a}^\dagger] = \frac{m\omega_0}{2\hbar}\left([\hat{x}, \hat{x}] - \frac{2i}{m\omega_0}[\hat{x}, \hat{p}] + \frac{1}{m^2\omega_0^2}[\hat{p}, \hat{p}]\right) = \frac{[\hat{x}, \hat{p}]}{i\hbar} = 1$$

7.9 Say $\hat{a}^\dagger|n\rangle = C_n|n+1\rangle$. Then the norm of this state is

$$\langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n|(\hat{a}^\dagger\hat{a} + 1)|n\rangle = \langle n|(\hat{N} + 1)|n\rangle = n + 1$$

Therefore, $C_n = \sqrt{n+1}$. Applying $\hat{a}^\dagger$ successively, we get

$$|n\rangle = \frac{1}{\sqrt{n!}}\hat{a}^{\dagger n}|0\rangle$$

This applies to the x-representation ϕ_n as well.

You deal with $\hat{a}$ the same way. say $\hat{a}|n\rangle = K_n|n-1\rangle$. Then,

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n$$

So $K_n = \sqrt{n}$.

7.15 Note that since $\hat{x} \propto \hat{a} + \hat{a}^\dagger$, $\langle m|\hat{x}|n\rangle \neq 0$ only for $m = n \pm 1$. Since $|\psi\rangle$ is a superposition of the 0th and 3rd levels, and $3 \neq 0 \pm 1$, $\langle x\rangle = 0$.

7.18 Work with energy eigenstates. Since all states can be written as a superposition of energy eigenstates, finding the minimum energy for energy eigenstates will give you the minimum for an arbitrary state as well.

Since (see previous problem) $\langle x\rangle = \langle p\rangle = 0$ in all energy eigenstates,

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle p^2\rangle}$$

In recitation problem 7.10 we found that $\langle V\rangle = \langle T\rangle$, so

$$\frac{1}{2}k\langle x^2\rangle = \frac{1}{2m}\langle p^2\rangle$$

meaning $\langle p^2 \rangle = mk\langle x^2 \rangle$ and hence $\Delta p = \sqrt{mk}\,\Delta x$. Now, the ground state energy is the sum of its kinetic and potential parts,

$$E_0 = k\langle x^2 \rangle = k(\Delta x)^2 = \frac{k\Delta x \Delta p}{\sqrt{mk}} \geq \omega_0 \frac{\hbar}{2}$$

So the minimum energy is at least $\frac{1}{2}\hbar\omega_0$, which is exactly what the ground state energy is.

7.30 With $\hat{x} \to -i\frac{\partial}{\partial k}$,

$$-i\frac{\partial}{\partial k}\varphi_x(k) = x\varphi_x(k)$$

It's obvious that solutions to this are the familiar plane waves,

$$\varphi_x(k) = e^{ikx}$$

where $k = p/\hbar$.