Homework Solutions # 8 (Liboff Ch. 9)

9.3 $L = \hbar\sqrt{l(l+1)}$, so $l = 7$ in our case. The x axis is arbitrary; the minimum angle will be the same as that for the z axis, where $L_z = \hbar m$. This will happen for the maximum $m = l = 7$. The angle is then $\theta = \cos^{-1}(m/\sqrt{l(l+1)}) = 21°$.

9.9

(a)

$$\left(\hat{L} \times \hat{r}\right)_x = \hat{L}_y \hat{z} - \hat{L}_z \hat{y} = \hat{z}\hat{L}_y + [\hat{L}_y, \hat{z}] - \hat{y}\hat{L}_z + [\hat{L}_z, \hat{y}] = -\left(\hat{r} \times \hat{L}\right)_x + 2i\hbar\hat{x}$$

generalizing to all components, this means

$$\hat{\mathbf{L}} \times \hat{\mathbf{r}} = 2i\hbar\hat{r} - \hat{\mathbf{r}} \times \hat{\mathbf{L}}$$

(b)

$$\left(\hat{L} \times \hat{r}\right)_x^\dagger - (i\hbar\hat{x})^\dagger = \hat{z}\hat{L}_y - \hat{y}\hat{L}_z + i\hbar\hat{x} = i\hbar\hat{x} - \left(\hat{r} \times \hat{L}\right)_x$$

Which is the same, from part (a).

(c) Breaking it apart, all the non-zero commutators are of the form

$$[\hat{L}_y^2, \hat{x}] = \hat{L}_y[\hat{L}_y, \hat{x}] + [\hat{L}_y, \hat{x}]\hat{L}_y = -i\hbar(\hat{L}_y\hat{z} + \hat{z}\hat{L}_y) = -i\hbar(2\hat{L}_y\hat{z} - i\hbar\hat{x})$$

So

$$[\hat{L}^2, \hat{x}] = -i\hbar(2\hat{L}_y\hat{z} - i\hbar\hat{x} - 2\hat{L}_z\hat{y} - i\hbar\hat{x}) = -2i\hbar\Theta_x$$

and so forth.

9.23 Using the ladder operators, and with a and b appropriate normalization constants,

$$\langle L_x \rangle = \langle lm|\frac{1}{2}(\hat{L}_+ + \hat{L}_-)|lm\rangle = a\langle lm|l, m+1\rangle + b\langle lm|l, m-1\rangle = 0$$

Since the eigenvectors are orthogonal. Same sort of calculation for $\langle L_y \rangle$.

From symmetry, $\langle L_x^2 \rangle = \langle L_y^2 \rangle$. So

$$\langle L_x^2 \rangle + \langle L_y^2 \rangle = 2\langle L_x^2 \rangle = \langle L^2 \rangle - \langle L_z^2 \rangle$$

Which means

$$\langle L_x^2 \rangle = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]$$

9.26 Note that

$$\langle \theta\phi|\varphi \rangle = \frac{i}{\sqrt{2}}\left(Y_1^1 + Y_1^{-1}\right)$$

So $|\varphi\rangle = \frac{i}{\sqrt{2}}(|11\rangle + |1\text{-}1\rangle)$.

(a) $L_z = \pm\hbar$, probability $\frac{1}{2}$ each.

(b) $\langle L_x \rangle = \frac{1}{2}\langle\varphi|(\hat{L}_+ + \hat{L}_-)|\varphi\rangle = 0$.

(c) $\langle L^2 \rangle = \hbar^2 1(1+1) = 2\hbar^2$.