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## Homework Solutions # 9 (Liboff Chapter 10)

**10.9** The space-dependent part of the wave function is real. Therefore  $\mathbf{J} = 0$  and so is its divergence.

### 10.10

- (a) Note that the radial part of the wavefunction is  $j_1(kr)$ ;  $\psi$  is an eigenfunction of the free particle Hamiltonian with energy  $E = \hbar^2 k^2 / 2m$ . So,

$$\psi(\mathbf{r}, t) = \psi(\mathbf{r}, 0)e^{-\frac{E}{\hbar}t}$$

- (b) Since it's an energy eigenstate, the expectation of the energy is the same  $E$  just calculated.
- (c)  $\psi$  is a  $\hat{L}^2$  eigenstate with  $l = 1$ . Therefore the only possible measured value is  $2\hbar^2$ .  $L_z$  measurements can find  $\frac{9}{34}N$  neutrons with  $L_z = 0$  and  $\frac{25}{34}N$  with  $L_z = -\hbar$
- (d) Since  $\psi$  is already an  $l = 1$  eigenstate, nothing will change upon measurement.
- (e) The  $m = -1$  subspace will be selected, so

$$\psi(\mathbf{r}, t) = 4\pi i j_1(kr) Y_1^{-1}(\theta, \phi) e^{-\frac{E}{\hbar}t}$$

**10.19** Using  $[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}$ ,

$$\begin{aligned} [\hat{x}_0, \hat{H}] &= \frac{1}{2m} \left[ \hat{x} + \frac{c}{eB} \hat{p}_y, \left( \hat{p}_x + \frac{eB}{c} \hat{y} \right)^2 + \hat{p}_y^2 + \hat{p}_z^2 \right] \\ &= \frac{1}{2m} \left\{ \left[ \hat{x}, \left( \hat{p}_x + \frac{eB}{c} \hat{y} \right)^2 \right] + \frac{c}{eB} \left[ \hat{p}_y, \left( \hat{p}_x + \frac{eB}{c} \hat{y} \right)^2 \right] \right\} \\ &= \frac{1}{2m} \left\{ [\hat{x}, \hat{p}_x^2] + \frac{2eBy}{c} [\hat{x}, \hat{p}_x] + 2\hat{p}_x [\hat{p}_y, \hat{y}] + \frac{eB}{c} [\hat{p}_y, \hat{y}^2] \right\} \\ &= \frac{1}{m} \left\{ i\hbar\hat{p}_x + \frac{eB}{c} i\hbar\hat{y} - i\hbar\hat{p}_x - \frac{eB}{c} i\hbar\hat{y} \right\} = 0 \end{aligned}$$

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and

$$[\hat{x}_0, \hat{y}_0] = -\frac{c}{eB}[\hat{x}, \hat{p}_x] = -\frac{i\hbar c}{eB} \neq 0$$

## 10.24

(a) Expressed in spherical coordinates,

$$\begin{aligned}\varphi &= Ar(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)e^{-k_0r} \\ &= 2\sqrt{\frac{\pi}{3}}Ar\left(e^{i\frac{3\pi}{4}}Y_1^1 + Y_1^0 + e^{i\frac{\pi}{4}}Y_1^{-1}\right)e^{-k_0r}\end{aligned}$$

Integrate over all of space to get

$$\int dr d\theta d\phi r^2 r^2 \sin\theta \varphi^2 = 1$$

So

$$\frac{1}{A^2} = \frac{4\pi}{3} \left( \int_{4\pi} d\Omega (|Y_1^1|^2 + |Y_1^0|^2 + |Y_1^{-1}|^2) \right) \left( \frac{1}{(2k_0)^5} \int_0^\infty d\rho \rho^4 e^{-\rho} \right) = \frac{3\pi}{k_0^5}$$

(b) We don't care about the radial part of  $\varphi$ ; only its overlap with  $Y_1^0$ . By inspection, each  $m$  value is equally probable;  $P = \frac{1}{3}$ .

(c) Integrate  $\varphi^2$  over the full solid angle ( $4\pi$ ) and over  $r < 1/k_0$ :

$$\frac{4}{3} \frac{1}{2^5} \int_0^2 d\rho \rho^4 e^{-\rho} = 1 - 7e^{-2} = 0.05$$

**10.31** The ground state has  $l = 0$ . In region I, the solution is a spherical Bessel function,  $j_0(kr)$ , with  $k$  defined as in the text. In region II, we get  $h_0^{(2)}(i\kappa r)$ . The overall normalization is unimportant for finding the eigenenergies, so we can just use one constant to set to join the two regions smoothly. The book uses  $-ih_0^{(2)}(i\kappa r)$  as the basic  $\phi_{II}$ , so let's stick with that.

$$\phi_I = \frac{\sin kr}{kr} \quad \phi_{II} = A \frac{e^{-\kappa r}}{\kappa r}$$

Continuity of the wavefunction and its derivative at  $r = a$  requires

$$\frac{\sin \xi}{\xi} = A \frac{e^{-\eta}}{\eta}$$

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$$\cos \xi - \frac{\sin \xi}{\xi} = -A \left( e^{-\eta} + \frac{e^{-\eta}}{\eta} \right)$$

Solving for  $A$  from the first equation and plugging it into the second, we get

$$\xi \cos \xi - \sin \xi = -(1 + \eta) \sin \xi \quad \Rightarrow \quad \eta = -\xi \cot \xi$$

A second constraint is from the definitions of  $\kappa$  and  $k$ :  $\xi^2 + \eta^2 = \rho^2$ . We can now calculate  $\eta = 0.54$ . Looking at the graph of curves in Figure 8.3, we can see that the curves intersect pretty low, so  $|V| \gg |E|$ . So a good approximation for  $-\xi \cot \xi$  is the first term in its Taylor series around  $\frac{\pi}{2}$ , which is  $\xi \approx \frac{2}{\pi}\eta + \frac{\pi}{2}$ . Doing the calculation, we get  $|V| = 30.9$  MeV, so  $|E|/|V| \approx 0.07$ .

Obviously, it is possible for a bound state not to exist, since even for this ground state, it is possible that the  $\xi^2 + \eta^2 = \rho^2$  quarter-circle does not intersect with any  $\eta = -\xi \cot \xi$  line.

### 10.38

- (a) Noticing that  $\psi = \frac{1}{\sqrt{2}}\varphi_{100} - i4\sqrt{3}A\varphi_{211} + 4\sqrt{3}A\varphi_{21-1} + 4\sqrt{21}A\varphi_{210}$  and that the eigenfunctions are orthonormal, we get

$$\int d^3r |\psi|^2 = \frac{1}{2} + 48A^2 + 48A^2 + 336A^2 = 1$$

giving  $A = 1/12\sqrt{6}$ .

- (b)  $P(l=0) = P(l=1) = \frac{1}{2}$ .  
 (c)  $P_r(r) = \int d\Omega r^2 |\psi|^2$ . Noting that  $Y_l^m$  are normalized over the full solid angle,

$$P_r(r) = \frac{r^2}{2a_0^3} e^{-2r/a_0} + \frac{r^4}{24a_0^5} e^{-r/a_0} \left( \frac{1}{18} + \frac{1}{18} + \frac{7}{18} \right) = \frac{r^2}{2a_0^3} e^{-2r/a_0} + \frac{r^4}{48a_0^5} e^{-r/a_0}$$

- (d) Setting the  $r$ -derivative to zero gives

$$4 - \frac{4r}{a_0} + \frac{r^2}{12a_0^2} e^{r/a_0} - \frac{r^3}{48a_0^3} e^{r/a_0} = 0$$

This has to be solved numerically, for  $r/a_0 = 1.048$ .

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(e) Two different energies – so with  $\omega_1 = -|E_1|/\hbar$  and  $\omega_2 = -|E_1|/4\hbar$ ,

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\varphi_{100}e^{-i\omega_1 t} + [-i4\sqrt{3}A\varphi_{211} + 4\sqrt{3}A\varphi_{21-1} + 4\sqrt{21}A\varphi_{210}] e^{-i\omega_2 t}$$

(f) Only one  $m = 1$  component, so

$$\psi(\mathbf{r}, t) = \varphi_{211}e^{-i\omega_2 t}$$

(g) Two  $m = 0$  components, so we need to re-normalize.

$$\psi(\mathbf{r}, t) = \sqrt{\frac{9}{8}} \left( \frac{1}{\sqrt{2}}\varphi_{100}e^{-i\omega_1 t} + \frac{\sqrt{7}}{3\sqrt{2}}\varphi_{210}e^{-i\omega_2 t} \right)$$

(h) The radial momentum operator affects only the radial part of the wavefunctions; the  $Y_l^m$ 's are orthonormal. So start with the  $R_{10}$  bit:

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r e^{-r/a_0}) = \left( \frac{1}{a_0^2} - \frac{2}{a_0 r} \right) e^{-r/a_0}$$

So

$$\int dr r^2 R_{10} \frac{1}{r} \frac{\partial^2}{\partial r^2} R_{10} = \frac{4}{a_0^4} \int dr \left( \frac{r^2}{a_0} - 2r \right) e^{-2r/a_0}$$

Changing variables to  $\rho = 2r/a_0$ , this becomes

$$\frac{4}{a_0^4} \left[ \frac{1}{a_0} \left( \frac{a_0}{2} \right)^3 \int d\rho \rho^2 e^{-\rho} - 2 \left( \frac{a_0}{2} \right)^2 \int d\rho \rho e^{-\rho} \right] = -\frac{1}{a_0^2}$$

using  $\int d\rho \rho^m e^{-\rho} = m!$ . Doing a similar second derivative and then integration with the  $R_{21}$  part, the expectation value becomes

$$\left\langle -\frac{p_r^2}{2\mu} \right\rangle = -\frac{\hbar^2}{2\mu} \left( \frac{1}{a_0^2} + \frac{1}{12a_0^2} \right) = -\frac{13}{12}|E_1|$$

Only  $\langle H \rangle$  remains:

$$\langle H \rangle = -\frac{1}{2}|E_1| - \frac{1}{2} \frac{|E_1|}{2^2} = -\frac{5}{8}|E_1|$$

So

$$\langle H_s \rangle = -\frac{41}{24}|E_1| = -23.2 \text{ eV}$$

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(i) The lowest energy is  $-|E_1| = -13.6$  eV.

**10.43** Noticing that the angular integrals are all 1 because the  $Y_l^m$  are orthonormal,

$$\left\langle \frac{1}{r} \right\rangle = \int_0^\infty dr r^2 \frac{1}{r} (2\kappa)^3 A_{nl}^2 \rho^{2l} e^{-\rho} F_{nl}^2(\rho) = 2\kappa A_{nl}^2 \int_0^\infty d\rho \rho^{2l+1} e^{-\rho} (L_{n-l-1}^{2l+1})^2(\rho)$$

Using the orthogonality relationship between associated Laguerre polynomials,

$$\left\langle \frac{1}{r} \right\rangle = \frac{2Z}{a_0 n} \frac{(n-l-1)!}{2n[(n+l)!]^3} \frac{[(n+l)!]^3}{(n-l-1)!} = \frac{Z}{a_0 n^2}$$

Using the definition of the Bohr radius,

$$\langle V(r) \rangle = -\frac{Z^2 e^2}{a_0 n^2} = -\frac{Z^2 \mu e^4}{\hbar^2 n^2}$$

To get  $\langle T \rangle$ , notice that  $\hat{H} = \hat{T} + \hat{V}$ . So

$$\langle T \rangle = \langle H \rangle - \langle V \rangle = -\frac{Z^2 \mu e^4}{2\hbar^2 n^2} + \frac{Z^2 \mu e^4}{\hbar^2 n^2} = \frac{Z^2 \mu e^4}{2\hbar^2 n^2}$$