Enumerating Rook and Queen Paths

Martin Erickson, Suren Fernando, and Khang Tran

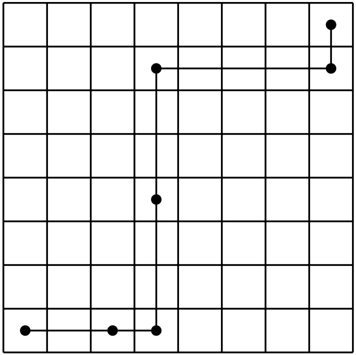
October 8, 2009

Abstract

How many ways can a chess Rook or Queen move from a corner cell to the opposite corner cell of an arbitrary size, arbitrary dimensional chess board, assuming that the piece moves toward the goal cell at each step? Recurrence relations, generating functions, and asymptotic formulas are given for the number of paths. Some open problems are presented.

Rook Paths

The Rook may move any number of squares to the right or up in one step.



Sum the Horizontal and Vertical Predecessors

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
64	320	1328	4864	16428	52356	159645	470010	...
32	144	560	1944	6266	19149	56190	159645	...
16	64	232	760	2329	6802	19149	52356	...
8	28	94	289	838	2329	6266	16428	...
4	12	37	106	289	760	1944	4864	...
2	5	14	37	94	232	560	1328	...
1	2	5	12	28	64	144	320	...
1	1	2	4	8	16	32	64	...

470010 paths from lower-left square to upper-right square

Recurrence Relation and Rational Generating Function

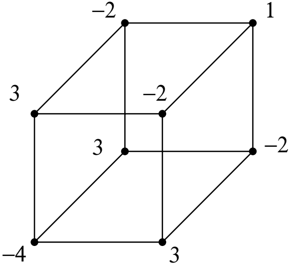
$a(0,0)=1,\ a(0,1)=1,\ a(1,0)=1,\ a(1,1)=2;$
$a(m,n)=2a(m-1,n)+2a(m,n-1)-3a(m-1,n-1),$
$m\geq 2$ or $n\geq 2$.

$$\begin{array}{rr} -2 & 1 \\ 3 & -2 \end{array}$$

$$\sum_{m\geq 0,\, n\geq 0} a(m,n)s^m t^n = \frac{1-s-t+st}{1-2s-2t+3st}.$$

3-D Rook

How many ways can a Rook move from $(0, 0, 0)$ to (m, n, o), where each step is a multiple of $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$?



$$\frac{(1-s)(1-t)(1-u)}{1-2(s+t+u)+3(st+su+tu)-4stu}.$$

Stamp Rolls Problem

3-Cent Stamp Roll:

$3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + \cdots$

5-Cent Stamp Roll:

$5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + \cdots$

How many ways can we make an amount n?

$56 = (3+3)+(3)+(5+5+5+5)+(3+3)+(5)+(5+5)+(3+3)$

$$\frac{(1 - x^3)(1 - x^5)}{1 - 2(x^3 + x^5) + 3(x^3 \cdot x^5)}$$

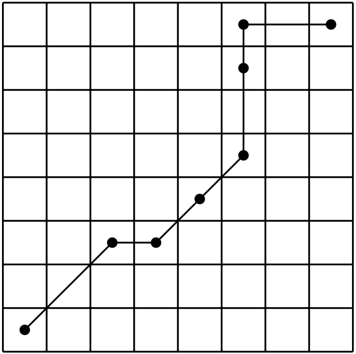
Lattice Paths Where Steps are Multiples of Basic Steps

$$\frac{\prod_{i=1}^{k}(1-x^{u_i})}{\sum_{j=0}^{k}(-1)^j(j+1)\sigma_j},$$

basic steps: $u_i = (u_{i1}, \ldots, u_{id})$, $1 \le i \le k$
$x^{\alpha} = x_1^{\alpha_1} \ldots x_k^{\alpha_k}$
σ_j = jth elementary symmetric polynomial in x^{u_i}

Queen Paths

The Queen may move any number of squares to the right or up or right-up in one step.



1499858 paths from lower-left square to upper-right square

Queen Paths Generating Function and Recurrence Relation

$$\frac{(1-x)(1-y)(1-xy)}{1-2(x+y+xy)+3(x\cdot y+x\cdot xy+y\cdot xy)-4(x\cdot y\cdot xy)}$$

$$\begin{array}{rrr} 0 & -2 & 1 \\ 3 & 1 & -2 \\ -4 & 3 & 0 \end{array}$$

Order and Degree of Recurrence Relations

Open Question: For d-dimensional Rook and Queen paths, what are the order and degree (of polynomial coefficients) of a linear recurrence relation satisfied by the diagonal sequence or the pure sequences (all coordinates but one fixed)?

2-D Rook Diagonal Sequence

$a_n = a(n,n)$
change of variables $t = x/s$ (so that $st = x$)
use Laurent series or residues:

$$f(x) = \frac{1}{2}\left(1 + \sqrt{\frac{1-x}{1-9x}}\right).$$

$$a_n \sim (\sqrt{2}/3)\, 9^n/\sqrt{\pi n}$$

$$f'(x)(1-x)(1-9x) - 4f(x) + 2 = 0$$

$$a_0 = 1,\ a_1 = 2;$$
$$a_n = ((10n-6)a_{n-1} - (9n-18)a_{n-2})/n, \quad n \geq 2.$$

Open Question: Is there a combinatorial proof of this recurrence formula?

2-D Queen, Diagonal Sequence

$b_n = b(n, n)$

$$f(x) = \frac{(x-1)}{(3x-2)}\left[1 + \frac{1-x}{\sqrt{1-12x+16x^2}}\right]$$

$$\begin{aligned}
&b_0 = 1,\ b_1 = 3,\ b_2 = 22,\ b_3 = 188;\\
&b_n = ((29n-18)b_{n-1} + (-95n+143)b_{n-2}\\
&+ (116n-302)b_{n-3} + (-48n+192)b_{n-4})/(2n), \quad n \geq 4.
\end{aligned}$$

$$b_n \sim c(1/r_1)^n/\sqrt{\pi n},$$

where $c = \sqrt{10(3\sqrt{5}-5)}/8,\ r_1 = (3-\sqrt{5})/8$

Recurrence Relations for Nim Games

Nim: Players take any number of stones from one pile of several piles (Rook paths)

Wythoff's Nim: Players take the same number of stones from any number of piles (Queen paths)

Number of games with equal piles of stones:

	order	degree
Nim		
2 piles	2	1
3 piles (empirical)	4	3
Wythoff's Nim		
2 piles	4	1